

THE \mathcal{LS} METHOD FOR THE CLASSICAL GROUPS IN POSITIVE CHARACTERISTIC AND THE RIEMANN HYPOTHESIS

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ABSTRACT. We provide a definition for an extended system of γ -factors for products of generic representations τ and π of split classical groups or general linear groups over a non-archimedean local field of characteristic p . We prove that our γ -factors satisfy a list of axioms (under the assumption $p \neq 2$ when both groups are classical groups) and show their uniqueness (in general). This allows us to define extended local L -functions and root numbers. We then obtain automorphic L -functions $L(s, \tau \times \pi)$, where τ and π are globally generic cuspidal automorphic representations of split classical groups or general linear groups over a global function field. In addition to rationality and the functional equation, we prove that our automorphic L -functions satisfy the Riemann Hypothesis.

INTRODUCTION

Let \mathbf{G}_m and \mathbf{G}_n denote either split classical groups or general linear groups of ranks m and n , respectively. Let k be a global function field with finite field of constants \mathbb{F}_q and ring of adèles \mathbb{A}_k . We present a theory of automorphic L -functions $L(s, \tau \times \pi)$, where τ and π are globally generic cuspidal automorphic representations of $\mathbf{G}_m(\mathbb{A}_k)$ and $\mathbf{G}_n(\mathbb{A}_k)$.

The case of $\mathbf{G}_m = \mathrm{GL}_m$ and \mathbf{G}_n a classical group is made possible by our work on the Langlands-Shahidi method in positive characteristic for the classical groups [10, 11]. Already in this case, the \mathcal{LS} method has particularly interesting applications. In addition to the Ramanujan conjecture for the classical groups over function fields established in [10], the zeros of $L(s, \tau \times \pi)$ satisfy $\Re(s) = 1/2$.

We note that the case of $\mathbf{G}_m = \mathrm{GL}_m$ and $\mathbf{G}_n = \mathrm{GL}_n$ gives rise to Rankin-Selberg factors. Indeed, we include a treatise of this case in a self contained manner within the Langlands-Shahidi method in [11]. And, we provide a short proof of the equality of local factors when defined via different methods in [6]. Thanks to the work of Lafforgue on the Langlands correspondence for GL_N over function fields, the Riemann Hypothesis is available for L -functions of products of cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A}_k)$ and $\mathrm{GL}_n(\mathbb{A}_k)$ [8]. In this very important case, all of our results are available with no restriction on the characteristic of k .

Notice that the case of two classical groups \mathbf{G}_m and \mathbf{G}_n is treated for the first time here in positive characteristic. We provide axioms for an extended system of γ -factors, L -functions and root numbers which cover all of the above mentioned cases for \mathbf{G}_m and \mathbf{G}_n . We first establish existence and uniqueness of γ -factors, Theorem 1.5, and then include existence and uniqueness of local L -functions and ε -factors in Theorem 4.3. The theory is complete under the assumption $\mathrm{char}(k) \neq 2$.

We begin by introducing notation that is useful when dealing with systems of γ -factors, L -functions and root numbers. Local factors $\gamma(s, \tau \times \pi, \psi)$, $L(s, \tau \times \pi)$ and $\varepsilon(s, \tau \times \pi, \psi)$ are defined on the local class $\mathfrak{ls}(p, \mathbf{G}_m, \mathbf{G}_n)$, while global L -functions and root numbers

$$L(s, \tau \times \pi) = \prod_v L(s, \tau_v \times \pi_v), \quad \varepsilon(s, \tau \times \pi) = \prod_v \varepsilon(s, \tau_v \times \pi_v, \psi_v)$$

are defined on the global class $\mathcal{LS}(p, \mathbf{G}_m, \mathbf{G}_n)$ (see § 1.1 and § 1.2). We often write $\mathfrak{ls}(p)$ and $\mathcal{LS}(p)$ when there is no need to specify the groups \mathbf{G}_m and \mathbf{G}_n . Theorem 4.4 can then be succinctly stated as follows:

Theorem. *Automorphic L -functions on $\mathcal{LS}(p)$, $p \neq 2$, satisfy the following properties:*

- (i) (Rationality). $L(s, \tau \times \pi)$ converges on a right half plane and has a meromorphic continuation to a rational function on q^{-s} .
- (ii) (Functional equation). $L(s, \tau \times \pi) = \varepsilon(s, \tau \times \pi) L(1 - s, \tilde{\tau} \times \tilde{\pi})$.
- (iii) (Riemann Hypothesis) The zeros of $L(s, \tau \times \pi)$ are contained in the line $\Re(s) = 1/2$.

Let us now give a more detailed description of the contents of the article. Theorem 1.5 concerns the existence and uniqueness of a system of γ -factors on $\mathfrak{ls}(p)$. The theorem is true with no assumption on p for $\mathfrak{ls}(p, \mathrm{GL}_m, \mathbf{G}_n)$. However, we assume $p \neq 2$ in order to produce γ -factors on $\mathfrak{ls}(p, \mathbf{G}_m, \mathbf{G}_n)$ when both \mathbf{G}_m and \mathbf{G}_n are classical groups. Extended γ -factors satisfy several local properties including a twin multiplicativity property when the representations are obtained via parabolic induction, as well as a stability property on $\mathfrak{ls}(p, \mathrm{GL}_1, \mathbf{G}_n)$ with respect to highly ramified characters of GL_1 . Globally, γ -factors make an important appearance in the functional equation for partial L -functions on $\mathcal{LS}(p)$.

All of § 2 is devoted to a proof of the existence part of Theorem 1.5. With the Langlands-Shahidi method now complete for the split classical groups in positive characteristic, we give a straightforward and linearly ordered presentation of γ -factors on $\mathfrak{ls}(p, \mathrm{GL}_m, \mathbf{G}_n)$. We recall the basic definitions in § 2.1. Then, our results on the Siegel Levi case for the split classical groups [11] are summarized in § 2.2. The Siegel Levi case allows us to define exterior and symmetric square γ -factors, which are uniquely characterized and are proved to be in accordance with the local Langlands conjecture for GL_m in [4]. Filling any gaps that were left in [10], the case of $\mathfrak{ls}(p, \mathrm{GL}_m, \mathbf{G}_n)$ is presented in § 2.3 with no assumption on the characteristic. The new case of $\mathfrak{ls}(p, \mathbf{G}_m, \mathbf{G}_n)$, with both \mathbf{G}_m and \mathbf{G}_n classical groups, is developed in § 2.4 under the assumption $p \neq 2$.

There are fundamental differences in local-to-global arguments between global fields of characteristic zero and characteristic p . In § 3, we use a variation of a result of Vignéras that allows us to lift a local pair of irreducible supercuspidal generic representations τ_0 and π_0 to a global pair of globally generic cuspidal automorphic representations τ and π (see Proposition 3.1). We remark that, over number fields, Shahidi makes a crucial improvement upon the local-to-global argument of Henniart and Vignéras by incorporating the archimedean theory that is available at infinite places (Proposition 5.1 of [15]). Over function fields, the main difference is due to the fact that all places are non-archimedean; a place at infinity plays the role of the archimedean places. As a further remark, in the case of two general linear groups

we have a much more precise local-to-global result [4], which allows us to remove stability from the list of properties required in the characterization [6].

The uniqueness part of Theorem 1.5 is proved in § 3 over a global function field with no restriction on p . We first treat the case of $\mathfrak{Is}(p, \mathrm{GL}_1, \mathbf{G}_n)$ in § 3.2, where we can use stability of γ -factors combined with the Grundwald-Wang theorem of class field theory. We then proceed to the general case in § 3.3. Our method of proof resembles the approach taken in [10], and we refer to the introduction for further remarks on the local-to-global argument over a global function field (see also § 3).

Building upon extended γ -factors, we axiomatize local L -functions and root numbers on $\mathfrak{Is}(p)$ in § 4.1. Additional properties of γ -factors are recorded in § 4.2, these include a local functional equation for which we provide a proof. We extend Theorem 1.5 to a theorem on extended γ -factors, local L -functions and root numbers in § 4.3. Finally, we establish our main global results in § 4.4, under the assumption $p \neq 2$. Theorem 4.4 includes rationality, the functional equation and the Riemann Hypothesis for automorphic L -functions on $\mathcal{LS}(p)$.

Our general results are possible since we have established a Langlands functorial lift from globally generic cuspidal automorphic representations π of a classical group \mathbf{G}_n to automorphic representations Π of GL_N [10]. The integer N is obtained from the table of § 1.3, by minimally embedding the connected component of the Langlands dual group ${}^L\mathbf{G}_n$ of \mathbf{G}_n into $\mathrm{GL}_N(\mathbb{C})$. The image of functoriality can be further expressed as an isobaric sum

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_d,$$

where each Π_i is a self-dual cuspidal automorphic representation of GL_{n_i} . They satisfy the additional condition that, for each i , a partial L -function $L^S(s, \Pi_i, r \circ \rho_{n_i})$ has a pole at $s = 1$, where $r = \wedge^2$ or Sym^2 depending on the classical group [17]. Exterior and symmetric square L -functions, and related γ -factors, are thoroughly studied in [4, 11]. Furthermore, [11] also develops the necessary theory for the Siegel Levi case of quasi-split Unitary groups. This leads us to Asai γ -factors and L -functions, which are uniquely characterized in [5]. They play a similar role for the quasi-split unitary groups, as the exterior and symmetric square L -functions do for the split classical groups, when describing the image of the functorial lift of [12] as an isobaric sum.

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1. EXTENDED γ -FACTORS

Let \mathbf{G}_n be either the general linear group GL_n or a split classical group of rank n . Given a ring R and an algebraic group \mathbf{G} defined over R , we often let G denote its group of rational points. Given a non-archimedean local field F , let \mathcal{O}_F denote its ring of integers, \mathfrak{p}_F its maximal ideal, ϖ_F a uniformizer, and q_F the cardinality of

its residue field. Given a global function field k , we let q denote the cardinality of its field of constants; for a place v of k , we let q_v be the cardinality of the residue field of k_v . Given a representation σ , we let $\tilde{\sigma}$ denote its contragredient.

1.1. Local notation. Let $\mathfrak{Is}(p, \mathbf{G}_m, \mathbf{G}_n)$ denote the class of quadruples (F, τ, π, ψ) consisting of: a non-archimedean local field F of characteristic p ; irreducible generic representations τ of $G_m = \mathbf{G}_m(F)$ and π of $G_n = \mathbf{G}_n(F)$; and, a non-trivial character ψ of F .

Given $(F, \tau, \pi, \psi) \in \mathfrak{Is}(p, \mathbf{G}_m, \mathbf{G}_n)$, we call it tempered (resp. discrete series, supercuspidal) if τ and π are tempered representations (resp. discrete series, supercuspidal).

1.2. Global notation. Let $\mathcal{LS}(p, \mathbf{G}_m, \mathbf{G}_n)$ denote the class of quadruples (k, τ, π, ψ) consisting of: a global function field k of characteristic p ; globally generic cuspidal automorphic representations $\tau = \otimes' \tau_v$ of $G_m = \mathbf{G}_m(\mathbb{A}_k)$ and $\pi = \otimes' \pi_v$ of $G_n = \mathbf{G}_n(\mathbb{A}_k)$; and, a non-trivial character $\psi = \otimes \psi_v$ of $k \backslash \mathbb{A}_k$.

Remark. We often write $\mathfrak{Is}(p)$ and $\mathcal{LS}(p)$ when there is no need to specify the groups \mathbf{G}_m and \mathbf{G}_n .

1.3. L -groups, principal series and partial L -functions. The connected components of the L -groups for the split classical groups are embedded into an appropriate dual group of GL_N according to the following table:

\mathbf{G}_n	${}^L G_n^0 \hookrightarrow {}^L \mathrm{GL}_N^0$	GL_N
SO_{2n+1}	$\mathrm{Sp}_{2n}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C})$	GL_{2n}
SO_{2n}	$\mathrm{SO}_{2n}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C})$	GL_{2n}
Sp_{2n}	$\mathrm{SO}_{2n+1}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n+1}(\mathbb{C})$	GL_{2n+1}

We include the possibility of $\mathbf{G}_n = \mathrm{GL}_n$, where we take $N = n$. Also, we let ρ_n denote the standard representation of $\mathrm{GL}_n(\mathbb{C})$.

Let $(F, \tau, \pi, \psi) \in \mathfrak{Is}(p, \mathbf{G}_m, \mathbf{G}_n)$ and assume that τ and π are unramified principal series. Then, the Satake parametrization gives semisimple conjugacy classes $\{A_\tau\}$ in ${}^L G_m^0 \hookrightarrow \mathrm{GL}_M(\mathbb{C})$ and $\{B_\pi\}$ in ${}^L G_n^0 \hookrightarrow \mathrm{GL}_N(\mathbb{C})$. Then, L -functions for unramified principal series representations are defined by

$$L(s, \tau \times \pi) = \frac{1}{\det(I - \rho_M(A_\tau) \otimes \rho_N(B_\pi) q_F^{-s})}.$$

Given $(k, \tau, \pi, \psi) \in \mathcal{LS}(p, \mathbf{G}_m, \mathbf{G}_n)$, we take S to be a finite set of places of k such that τ , π and ψ are unramified for $v \notin S$. The corresponding partial L -functions are defined by

$$L^S(s, \tau \times \pi) = \prod_{v \notin S} L(s, \tau_v \times \pi_v).$$

We can show that partial L -functions converge on a right half plane; in fact, they have a meromorphic continuation to a rational function on q^{-s} .

1.4. Axioms for a system of γ -factors. The Langlands-Shahidi method in positive characteristic allows us to produce a system of rational functions $\gamma(s, \tau \times \pi, \psi) \in \mathbb{C}(q_F^{-s})$ on $\mathfrak{Is}(p, \mathrm{GL}_m, \mathbf{G}_n)$. In this article, we concoct a system of extended γ -factors on $\mathfrak{Is}(p, \mathbf{G}_m, \mathbf{G}_n)$. Extended γ -factors can be characterized by a list of local properties together with their role in the global functional equation.

- (i) (Naturality). *Let $(F, \tau, \pi, \psi) \in \mathfrak{Is}(p)$, and let $\eta : F' \rightarrow F$ be an isomorphism of local fields. Let $(F', \tau', \pi', \psi') \in \mathfrak{Is}(p)$ be the quadruple obtained from (F, τ, π, ψ) via η . Then*

$$\gamma(s, \tau \times \pi, \psi) = \gamma(s, \tau' \times \pi', \psi').$$

- (ii) (Isomorphism). *Let $(F, \tau, \pi, \psi) \in \mathfrak{Is}(p)$. If $(F, \tau', \pi', \psi) \in \mathfrak{Is}(p)$ is such that $\tau \simeq \tau'$ and $\pi \simeq \pi'$, then*

$$\gamma(s, \tau \times \pi, \psi) = \gamma(s, \tau' \times \pi', \psi).$$

- (iii) (Compatibility with class field theory). *Let $(F, \chi_1, \chi_2, \psi) \in \mathfrak{Is}(p, \mathrm{GL}_1, \mathrm{GL}_1)$. Then*

$$\gamma(s, \chi_1 \times \chi_2, \psi) = \gamma(s, \chi_1 \chi_2, \psi).$$

- (iv) (Multiplicativity). *Let $(F, \tau_i, \pi_j, \psi) \in \mathfrak{Is}(p, \mathbf{G}_{m_i}, \mathbf{G}_{n_j})$, $0 \leq i \leq d$, $0 \leq j \leq e$; \mathbf{G}_{m_0} and \mathbf{G}_{n_0} can be classical groups or general linear groups; $\mathbf{G}_{m_i} = \mathrm{GL}_{m_i}$ and $\mathbf{G}_{n_j} = \mathrm{GL}_{n_j}$ for $1 \leq i \leq d$, $1 \leq j \leq e$. Set $m = \sum m_i$ and $n = \sum n_j$. If \mathbf{G}_{m_0} (resp. \mathbf{G}_{n_0}) is a classical group, take \mathbf{G}_m (resp. \mathbf{G}_n) to be a classical group of the same type. Let \mathbf{P}_m (resp. \mathbf{P}_n) be the standard parabolic subgroup of \mathbf{G}_m (resp. \mathbf{G}_n) with Levi $\mathbf{M}_m = \prod_{i=1}^d \mathrm{GL}_{m_i} \times \mathbf{G}_{m_0}$ (resp. $\mathbf{M}_n = \prod_{j=1}^e \mathrm{GL}_{n_j} \times \mathbf{G}_{n_0}$). First, assume $m_0 \geq 1$ and $n_0 \geq 1$. Let τ be the generic constituent of*

$$\mathrm{ind}_{\mathbf{P}_m}^{G_m}(\tau_1 \otimes \cdots \otimes \tau_d \otimes \tau_0),$$

and let π be the generic constituent of

$$\mathrm{ind}_{\mathbf{P}_n}^{G_n}(\pi_1 \otimes \cdots \otimes \pi_e \otimes \pi_0).$$

When $m_0 = 0$ (resp. $n_0 = 0$) we make the following conventions: take τ_0 (resp. π_0) to be the trivial character of $\mathrm{GL}_1(F)$ if \mathbf{G}_m (resp. \mathbf{G}_n) is a symplectic group; in all other cases, we interpret γ -factors involving τ_0 (resp. π_0) to be trivial.

- (iv.a) *If both \mathbf{G}_m and \mathbf{G}_n are classical groups, then*

$$\begin{aligned} \gamma(s, \tau \times \pi, \psi) &= \gamma(s, \tau_0 \times \pi_0) \\ &\times \prod_{i=1}^d \gamma(s, \tau_i \times \pi_0, \psi) \gamma(s, \tilde{\tau}_i \times \pi_0, \psi) \prod_{j=1}^e \gamma(s, \tau_0 \times \pi_j, \psi) \gamma(s, \tau_0 \times \tilde{\pi}_j, \psi) \\ &\times \prod_{1 \leq h \leq d, 1 \leq l \leq e} \gamma(s, \tau_h \times \pi_l, \psi) \gamma(s, \tau_h \times \tilde{\pi}_l, \psi) \gamma(s, \tilde{\tau}_h \times \pi_l, \psi) \gamma(s, \tilde{\tau}_h \times \tilde{\pi}_l, \psi). \end{aligned}$$

- (iv.b) *If $\mathbf{G}_m = \mathrm{GL}_m$ and \mathbf{G}_n is a classical group, then*

$$\gamma(s, \tau \times \pi, \psi) = \prod_{i=0}^d \gamma(s, \tau_i \times \pi_0, \psi) \times \prod_{i=1}^d \prod_{j=1}^e \gamma(s, \tau_i \times \pi_j, \psi) \gamma(s, \tau_i \times \tilde{\pi}_j, \psi).$$

(iv.c) If $\mathbf{G}_m = \mathrm{GL}_m$ and $\mathbf{G}_n = \mathrm{GL}_n$, then

$$\gamma(s, \tau \times \pi, \psi) = \prod_{i,j} \gamma(s, \tau_i \times \pi_j, \psi).$$

(v) (Dependence on ψ). Let $(F, \tau, \pi, \psi) \in \mathfrak{ls}(p, \mathbf{G}_m, \mathbf{G}_n)$. Given $a \in F^\times$, let ψ^a denote the character of F given by $\psi^a(x) = \psi(ax)$. Then

$$\gamma(s, \tau \times \pi, \psi^a) = \omega_\tau(a)^h \omega_\pi(a)^l |a|_F^{hl(s-\frac{1}{2})} \gamma(s, \tau \times \pi, \psi),$$

where $h = 2m$ if $\mathbf{G}_m = \mathrm{SO}_{2m}, \mathrm{SO}_{2m+1}$; $h = 2m + 1$ if $\mathbf{G}_m = \mathrm{Sp}_{2m}$; $h = m$ if $\mathbf{G}_m = \mathrm{GL}_m$; and, similarly for l , depending on \mathbf{G}_n .

(vi) (Stability). Let $(F, \eta, \pi_i, \psi) \in \mathfrak{ls}(p, \mathrm{GL}_1, \mathbf{G}_n)$, $i = 1, 2$, where π_1 and π_2 have the same central character and η is highly ramified. Then

$$\gamma(s, \eta \times \pi_1, \psi) = \gamma(s, \eta \times \pi_2, \psi).$$

(vii) (Functional equation). Let $(k, \tau, \pi, \psi) \in \mathcal{LS}(p)$, then

$$L^S(s, \tau \times \pi) = \prod_{v \in S} \gamma(s, \tau_v \times \pi_v, \psi_v) L^S(1 - s, \tilde{\tau} \times \tilde{\pi}).$$

1.5. Theorem. *There exists a system of γ -factors on $\mathfrak{ls}(p, \mathrm{GL}_m, \mathbf{G}_n)$ satisfying properties (i) – (vii). If $p \neq 2$, there exists a system of γ -factors on $\mathfrak{ls}(p, \mathbf{G}_m, \mathbf{G}_n)$ satisfying properties (i) – (vii). Any system of γ -factors satisfying properties (i) – (vii) is uniquely determined.*

2. EXISTENCE

A treatise of γ -factors, L -functions and root numbers for general linear groups is presented in a self contained manner within the Langlands-Shahidi method in [11] and the appendix [6]. We now complete the study begun in [10] for the cases involving split classical groups.

2.1. The Langlands-Shahidi local coefficient for the split classical groups in positive characteristic. Let \mathbf{G} be a split classical group of rank l and let $\mathbf{B} = \mathbf{T}\mathbf{U}$ be the Borel subgroup of upper triangular matrices with maximal torus \mathbf{T} and unipotent radical \mathbf{U} . Let \mathbf{P} be the standard maximal parabolic subgroup of \mathbf{G} with maximal Levi $\mathbf{M} = \mathrm{GL}_m \times \mathbf{G}_n$, $l = m + n$, and unipotent radical \mathbf{N} . Given $(F, \tau, \pi, \psi) \in \mathfrak{ls}(p, \mathrm{GL}_m, \mathbf{G}_n)$, we can form a smooth irreducible generic representation $\sigma = \tau \otimes \tilde{\pi}$ of M .

Let Σ denote the roots of \mathbf{G} with respect to the maximal torus \mathbf{T} , Σ^+ the positive roots, and Δ a basis fixed by our choice of Borel subgroup. Let \mathbf{N}_α denote the one parameter subgroup associated to $\alpha \in \Sigma$. The surjection $\mathbf{N} \twoheadrightarrow \mathbf{N} / \prod_{\alpha \in \Sigma^+ - \Delta} \mathbf{N}_\alpha \simeq \prod_{\alpha \in \Delta} \mathbf{N}_\alpha$ allows us to extend the character ψ of F to a non-degenerate character ψ of N by setting $\psi(\sum_{\alpha \in \Delta} x_\alpha) = \prod_{\alpha \in \Delta} \psi(x_\alpha)$. Let $\alpha_m \in \Delta$ be such that \mathbf{P} is the maximal parabolic subgroup corresponding to $\Delta - \{\alpha_m\}$. Then, we let ψ_M be the character of $U_M = U \cap M$ obtained from ψ via the surjection $\mathbf{U}_M \twoheadrightarrow \prod_{\alpha \in \Delta - \{\alpha_m\}} \mathbf{N}_\alpha$.

For every $\alpha \in \Delta$, we fix an embedding for the corresponding semisimple rank one group \mathbf{G}_α into \mathbf{G} and fix a representative w_α for the corresponding Weyl group element as in [11]. We abuse notation and identify Weyl group elements with their fixed representatives. Let $w_0 = w_l w_{l,M}$, where w_l is the longest Weyl group element of \mathbf{G} and $w_{l,M}$ is the longest Weyl group element with respect to

M. Then, the non-degenerate characters ψ of N and ψ_M of U_M are w_0 -compatible, i.e., $\psi(u) = \psi_M(w_0^{-1}uw_0)$ for $u \in U_M$.

Let $\sigma = \tau \otimes \tilde{\pi}$ be a ψ_M -generic representation of $M = \mathrm{GL}_m(F) \times G_n$. We then let $I(s, \sigma)$ be the unitarily induced representation

$$\mathrm{ind}_P^{G_l}(|\det(\cdot)|_F^s \tau \otimes \tilde{\pi}).$$

Let $V(s, \sigma)$ denote the space of $I(s, \sigma)$. If λ_M is a Whittaker functional for σ , then $I(s, \sigma)$ is ψ -generic for the Whittaker functional λ_ψ given by

$$\lambda_\psi(s, \sigma)f = \int_N \lambda_M(w_0^{-1}n) \overline{\psi}(n) dn,$$

where $f \in V(s, \sigma)$. The integral on the right hand side converges as a principal value integral over compact open subgroups of N .

We also have an intertwining operator $A(s, \sigma, w_0) : V(s, \sigma) \rightarrow V(-s, w_0^{-1}(\sigma))$, given by

$$A(s, \sigma, w_0)f(g) = \int_N f(w_0^{-1}ng) dn,$$

where we write $w_0(\sigma)$ for the representation given by $w_0(\sigma)(x) = \sigma(w_0^{-1}xw_0)$. It converges for $\Re(s) \gg 0$ and extends to a rational operator on q_F^{-s} .

The local coefficient is then defined using the uniqueness property of Whittaker models and the relationship

$$\lambda_\psi(s, \sigma)f = C_\psi(s, \sigma, w_0)\lambda_\psi(-s, w_0(\sigma))A(s, \sigma, w_0)f.$$

The local coefficient $C_\psi(s, \sigma, w_0)$ is a rational function on q_F^{-s} .

2.2. The case of a Siegel Levi subgroup. The case of a Siegel Levi subgroup $\mathbf{M} \simeq \mathrm{GL}_n$ of \mathbf{G}_n is studied in [4] when $\mathbf{G}_n = \mathrm{SO}_{2n}$ or SO_{2n+1} . In these cases the Langlands-Shahidi method allows us to study exterior square and symmetric square L -functions; the case of a Siegel Levi subgroup of Sp_{2n} is included in [11]. These cases provide an important step in the Langlands-Shahidi method for the split classical groups. Given an irreducible generic representation τ of $\mathrm{GL}_n(F)$, we define

$$C_\psi(s, \tau, w_0) = \begin{cases} \gamma(s, \tau, \mathrm{Sym}^2 \rho_n, \psi) & \text{if } \mathbf{G} = \mathrm{SO}_{2n+1} \\ \gamma(s, \tau, \wedge^2 \rho_n, \psi) & \text{if } \mathbf{G} = \mathrm{SO}_{2n} \end{cases}.$$

Here, ρ_n is the standard representation of $\mathrm{GL}_n(\mathbb{C})$, the dual group of GL_n , and $\gamma(s, \tau, \psi)$ is a Godement-Jacquet γ -factor. For unramified principal series representations, the above definition agrees with the Satake parametrization and provides a definition of exterior square and symmetric square local factors for general smooth representations. We now state the main result of [4], which shows that Langlands-Shahidi γ -factors are in accordance with the local Langlands correspondence in positive characteristic [9].

Theorem. *Let τ be an irreducible smooth representation of $\mathrm{GL}_n(F)$ and let σ be an n -dimensional Frobenius-semisimple ℓ -adic representation of the Weil-Deligne group in the isomorphism class $\sigma(\tau)$ corresponding to τ via the local Langlands correspondence. Then*

$$\gamma(s, \tau, r \circ \rho_n, \psi) = \gamma(s, r \circ \sigma, \psi),$$

where $r = \mathrm{Sym}^2$ or \wedge^2 .

For the remaining case of a Siegel Levi subgroup of a split classical group, let $\gamma(s, \tau, \psi)$ denote a Godement-Jacquet γ -factor. Then

$$C_\psi(s, \tau, w_0) = \gamma(s, \tau, \psi) \gamma(2s, \tau, \wedge^2 \rho_n, \psi) \text{ if } \mathbf{G} = \mathrm{Sp}_{2n}.$$

2.3. The case of $\mathfrak{ls}(p, \mathrm{GL}_m, \mathbf{G}_n)$. The study of γ -factors, L -functions and root numbers on $\mathfrak{ls}(p, \mathrm{GL}_m, \mathbf{G}_n)$ and $\mathcal{LS}(p, \mathrm{GL}_m, \mathbf{G}_n)$ was begun in [10]. We now gather the necessary results that establish the existence part of Theorem 1.5 in these cases; we use the conventions of [11] regarding Weyl group element representatives and normalization of Haar measures. Let $(F, \tau, \pi, \psi) \in \mathfrak{ls}(p, \mathrm{GL}_m, \mathbf{G}_n)$ and let $\sigma = \tau \otimes \tilde{\pi}$. We first assume that σ is a ψ_M -generic representation of M .

Having defined exterior and symmetric square γ -factors, which are in accordance with the local Langlands correspondence for GL_n , the γ -factors $\gamma(s, \tau \times \pi, \psi)$ are defined via the local coefficient:

$$C_\psi(s, \sigma, w_0) = \begin{cases} \gamma(s, \tau \times \pi, \psi) \gamma(2s, \tau, \mathrm{Sym}^2 \rho_n, \psi) & \text{if } \mathbf{G} = \mathrm{SO}_{2n+1} \\ \gamma(s, \tau \times \pi, \psi) \gamma(2s, \tau, \wedge^2 \rho_n, \psi) & \text{if } \mathbf{G} = \mathrm{Sp}_{2n} \text{ or } \mathrm{SO}_{2n} \end{cases}.$$

Let $(F, \tau, \pi, \psi) \in \mathfrak{ls}(p, \mathrm{GL}_m, \mathbf{G}_n)$, with $\sigma = \tau \otimes \tilde{\pi}$ ψ_M -generic. An isomorphism of local fields $\eta : F' \rightarrow F$, takes normalized Haar measures on $\mathbf{N}(F)$ to normalized Haar measures on $\mathbf{N}(F')$. Hence, the local coefficients $C_\psi(s, \sigma, w_0)$ and $C_{\psi'}(s, \sigma', w_0)$ are equal. Also, we have an isomorphism property for the local coefficient given two isomorphic ψ_M -generic representations σ and σ' of M . Thus, properties (i) and (ii) follow for ψ_M -generic σ and σ' . Property (iii) is included in the list of semisimple rank one cases of [11]. Property (iv.b) for the classical groups is Theorem 6.7 of [10], where it is explicitly stated.

We now discuss the relationship for γ -factors as the character varies. Given a triple $(F, \tau, \pi, \psi) \in \mathfrak{ls}(p, \mathrm{GL}_m, \mathbf{G}_n)$, the representation $\sigma = \tau \otimes \tilde{\pi}$ is generic with respect to a non-degenerate character χ_M of U_M . We will take χ to be a non-degenerate character of U , which is w_0 -compatible with χ_M . Now, given the group \mathbf{G} , we can embed it into a group $\tilde{\mathbf{G}}$ with Borel subgroup $\tilde{\mathbf{B}} = \tilde{\mathbf{T}}\mathbf{U}$. The group $\tilde{\mathbf{G}}$ has the same derived group as \mathbf{G} and has $H^1(Z_{\tilde{\mathbf{G}}}) = \{1\}$ (see § 5 of [16]). Thus, \tilde{T} acts transitively on the set of non-degenerate characters of U . Let $t \in \tilde{T}$ be such that the non-degenerate character ψ_M of U_M is equal to $\chi_{t,M}$, where $\chi_{t,M}(u) = \chi_M(t^{-1}ut)$. The character χ is taken so that $\psi = \chi_t$ on U , and w_0 -compatibility is preserved for the action of $t \in \tilde{T}$ on the non-degenerate characters. Let σ_t be defined by $\sigma_t(m) = \sigma(t^{-1}mt)$. It is a ψ_M -generic representation. Then, we have γ -factors defined on all of $\mathfrak{ls}(p, \mathrm{GL}_m, \mathbf{G}_n)$ via the local coefficient as follows

$$\gamma(s, \pi, \psi) = C_\psi(s, \pi_t, w_0).$$

We have a formula for the local coefficient when the character ψ varies. Written explicitly for γ -factors gives property (v) on $\mathfrak{ls}(p, \mathrm{GL}_m, \mathbf{G}_n)$.

Property (vi), stability of γ -factors in positive characteristic, is the content of Theorem 6.12 of [10] for the split classical groups. This includes the case of characteristic 2.

Finally, let $(F, \tau, \pi, \psi) \in \mathcal{LS}(p, \mathrm{GL}_m, \mathbf{G}_n)$. The crude functional equation for the local coefficient of split classical groups over function fields, Theorem 5.14 of [loc. cit.], reads

$$L^S(s, \tau \times \pi) L^S(2s, \tau, r \circ \rho_m) = \prod_{v \in S} C_{\psi_v}(s, \tau_v \otimes \tilde{\pi}_v, w_0) L^S(1-s, \tilde{\tau} \times \tilde{\pi}) L^S(1-2s, \tilde{\tau}, r \circ \rho_m),$$

where $r = \text{Sym}^2$ or \wedge^2 depending on the classical group and we use the conventions of [11]. For every $v \in S$ we have that

$$C_{\psi_v} = \gamma(s, \tau_v \times \pi_v, \psi_v) \gamma(2s, \tau_v, r \circ \rho_m, \psi_v).$$

The study of exterior and symmetric square γ -factors begun in [10] is completed in [4, 11]. We have the functional equation:

$$L^S(s, \tau, r \circ \rho_m) = \prod_{v \in S} \gamma(s, \tau_v, r \circ \rho_m, \psi_v) L^S(1-s, \tilde{\tau}, r \circ \rho_m).$$

Combining this equation with the crude functional equation of the local coefficient gives property (vii) for the corresponding γ -factors:

$$L^S(s, \tau \times \pi) = \prod_{v \in S} \gamma(s, \tau_v \times \pi_v, \psi_v) L^S(1-s, \tilde{\tau} \times \tilde{\pi}).$$

Therefore, we can conclude that the existence part of Theorem 1.5 holds in the case of $\mathfrak{Is}(p, \text{GL}_m, \mathbf{G}_n)$. Notice that no assumption on the characteristic is made for this part of the theorem.

2.4. The general case. A system of γ -factors on $\mathfrak{Is}(p, \text{GL}_m, \mathbf{G}_n)$ gives a system of γ -factors on $\mathfrak{Is}(p, \mathbf{G}_m, \text{GL}_n)$ via the relationship

$$\gamma(s, \pi \times \tau, \psi) := \gamma(s, \tau \times \pi, \psi),$$

for $(F, \pi, \tau, \psi) \in \mathfrak{Is}(p, \mathbf{G}_m, \text{GL}_n)$. We now build upon § 9 of [10], which is written under the assumption $p \neq 2$. We make this assumption for the rest of this section. Also, we focus on the case of two classical groups \mathbf{G}_m and \mathbf{G}_n ; we let M and N denote the positive integers obtained from m and n via the table on § 1.3.

First, let $(F, \tau, \pi, \psi) \in \mathfrak{Is}(p, \mathbf{G}_m, \mathbf{G}_n)$ be such that τ is supercuspidal. By Proposition 9.4 of [10], there exists a generic representation T of GL_M such that

$$\gamma(s, \tau \times \rho, \psi) = \gamma(s, T \times \rho, \psi),$$

for every supercuspidal representation ρ of $\text{GL}_r(F)$. The representation T is unique due to Théorème 1.1 of [3]; it is called the local functorial lift of τ .

An irreducible generic discrete series representation τ of a classical group can be described in terms of its inducing data

$$(2.1) \quad \tau \hookrightarrow \text{ind}_{P_m}^{G_m} (\delta_1 \otimes \cdots \otimes \delta_d \otimes \delta'_1 \otimes \cdots \otimes \delta'_e \otimes \tau_0),$$

where the δ_i 's and the δ'_j 's are essentially square integrable representations of $\text{GL}_{m_i}(F)$ and τ_0 is an irreducible generic supercuspidal representation of G_{m_0} , with \mathbf{G}_{m_0} a classical group of the same type as \mathbf{G}_m . Following the results of Mœglin-Tadić [13], this is made precise in equation (9.1) of [10]. The case of $m_0 = 0$ is allowed, with appropriate interpretations for the corresponding formulas. If T_0 is the local functorial lift of τ_0 , then the local functorial lift T of τ is the generic constituent of

$$(2.2) \quad \text{ind}_{P_m}^{G_m} (\delta_1 \otimes \cdots \otimes \delta_d \otimes \delta'_1 \otimes \cdots \otimes \delta'_e T_0 \otimes \tilde{\delta}'_e \otimes \cdots \otimes \tilde{\delta}'_1 \otimes \tilde{\delta}_d \cdots \otimes \tilde{\delta}_1).$$

The representation T is a self-dual tempered representation of $\text{GL}_M(F)$.

Now, an irreducible generic tempered representation of G_m is a constituent of

$$(2.3) \quad \text{ind}_{P_m}^{G_m} (\delta_1 \otimes \cdots \otimes \delta_d \otimes \tau_0),$$

where the δ_i 's are discrete series representations of $\mathrm{GL}_{m_i}(F)$ and τ_0 is a generic discrete series of G_{m_0} , where \mathbf{G}_{m_0} is a classical group of the same kind as \mathbf{G}_m . Then, the local functorial lift T of τ is given by

$$(2.4) \quad \mathrm{ind}_{P_m}^{G_m}(\delta_1 \otimes \cdots \otimes \delta_d \otimes T_0 \otimes \tilde{\delta}_d \otimes \tilde{\delta}_1),$$

where T_0 is the local functorial lift of the generic discrete series representation τ_0 .

An arbitrary irreducible generic representation τ of G_m can be described via the work of Muić on the standard module conjecture [14]. Then,

$$(2.5) \quad \tau = \mathrm{ind}_{P_m}^{G_m}(\tau_1 \nu^{r_1} \otimes \cdots \otimes \tau_d \nu^{r_d} \otimes \tau_0).$$

Here, each τ_i is a tempered representation of $\mathrm{GL}_{m_i}(F)$; τ_0 is a generic tempered representation of G_{m_0} , where \mathbf{G}_{m_0} is a classical group of the same kind as \mathbf{G}_m ; and, $\nu = |\det(\cdot)|_F$. If $\mathbf{G}_m = \mathrm{SO}_{2n}$, and τ_0 is the trivial representation of $\mathbf{G}_{m_0}(F)$ and $m_d = 1$, the above formula needs the following modification

$$(2.6) \quad \tau = \mathrm{ind}_{P_m}^{G_m}(\tau_1 \nu^{r_1} \otimes \cdots \otimes \tau_d \nu^{r_d}),$$

where we have $0 < |r_d| < r_{d-1} < \cdots < r_1$. In all other cases, it is given by equation (2.5), where the exponents can be taken so that $0 < r_d < \cdots < r_1$. The local functorial lift T of τ is then given as the unique irreducible quotient of

$$(2.7) \quad \mathrm{ind}_{P_M}^{\mathrm{GL}_M(F)}(\tau_1 \nu^{r_1} \otimes \cdots \otimes \tau_d \nu^{r_d} \otimes T_0 \otimes \tilde{\tau}_d \nu^{-r_d} \otimes \cdots \otimes \tilde{\tau}_1 \nu^{r_1}),$$

where T_0 is the local functorial lift of τ_0 , with appropriate modifications if the induced representation is given by (2.6). The local lift has the property that

$$\gamma(s, \tau \times \rho, \psi) = \gamma(s, T \times \rho, \psi),$$

for any irreducible generic representation ρ of $\mathrm{GL}_r(F)$.

With a description of the local image of functoriality, we can now obtain a system of extended γ -factors. Given $(F, \tau, \pi, \psi) \in \mathfrak{Is}(p, \mathbf{G}_m, \mathbf{G}_n)$ let $(F, T, \pi, \psi) \in \mathfrak{Is}(p, \mathrm{GL}_M, \mathbf{G}_n)$ be such that T is the local functorial lift of τ . Then, we define

$$(2.8) \quad \gamma(s, \tau \times \pi, \psi) := \gamma(s, T \times \pi, \psi).$$

We have that $\gamma(s, \tau \times \pi, \psi) = \gamma(s, \pi \times \tau, \psi)$ for every $(F, \tau, \pi, \psi) \in \mathfrak{Is}(p, \mathbf{G}_m, \mathbf{G}_n)$. Furthermore, if $(F, T, \Pi, \psi) \in \mathfrak{Is}(p, \mathrm{GL}_M, \mathrm{GL}_N)$, where T and Π are the local functorial images of τ and π , then

$$\gamma(s, \tau \times \pi, \psi) = \gamma(s, T \times \Pi, \psi).$$

It is now an exercise to show that properties (i) and (ii) are verified; property (iii) remains as before; and, our definition is indeed compatible with multiplicativity, property (iv). The dependence on ψ can now be obtained from the corresponding property for $\gamma(s, T \times \Pi, \psi)$ (see property (iv) in the main Theorem of [6]). And, stability remains as before.

Given $(k, \tau, \pi, \psi) \in \mathcal{LS}(p, \mathbf{G}_m, \mathbf{G}_n)$ let T and Π be the functorial lifts of τ and π . This is possible via Theorem 9.1 of [10]. In fact, the work of Ginzburg, Rallis and Soudry allows us to give a precise description of the image of functoriality [17]. The global functorial lift T of τ can be expressed as an isobaric sum

$$(2.9) \quad T = T_1 \boxplus \cdots \boxplus T_d,$$

where each T_i , $1 \leq i \leq d$, is a unitary self-dual cuspidal automorphic representation of $\mathrm{GL}_{M_i}(\mathbb{A}_k)$. Also, $\Pi_i \not\cong \Pi_j$ whenever $i \neq j$. Furthermore, if S is a sufficiently large finite set of places of k , then

- (i) $L^S(s, \Pi_i, \wedge^2 \rho_m)$ has a pole at $s = 1$, if $\mathbf{G}_m = \mathrm{SO}_{2m+1}$;
- (ii) $L^S(s, \Pi_i, \mathrm{Sym}^2 \rho_m)$ has a pole at $s = 1$, if $\mathbf{G}_m = \mathrm{SO}_{2m}$ or Sp_{2m} .

We can similarly express the global functorial lift Π of π as an isobaric sum.

The functional equation for extended γ -factors can then be obtained from the above description of the global image and the functional equation for Rankin-Selberg γ -factors, i.e.,

$$\begin{aligned} L^S(s, \tau \times \pi) &= L^S(s, \mathrm{T} \times \Pi) \\ &= \prod_{v \in S} \gamma(s, \mathrm{T}_v \times \Pi_v, \psi_v) L^S(1-s, \tilde{\mathrm{T}} \times \tilde{\Pi}) \\ &= \prod_{v \in S} \gamma(s, \tau_v \times \pi_v, \psi_v) L^S(1-s, \tau \times \pi), \end{aligned}$$

for every $(k, \tau, \pi, \psi) \in \mathcal{LS}(p, \mathbf{G}_m, \mathbf{G}_n)$.

3. UNIQUENESS

In the cases involving classical groups, we use a variation of a local-to-global result of Vignéras, which follows from the proof of Theorem 2.2 of [18]. We note that, over a global function field, a place at infinity plays the role that archimedean places play over number fields; the notion of an automorphic representation over function fields is independent of the choice of place at infinity. To prove the following proposition, we start with a local field F and take a global field k such that $k_{v_0} \simeq F$ at a place v_0 of k . We fix two different places v_1 and v_2 over the same function field k . Then one can apply the observation made on p. 469 of [loc. cit.] to the construction of globally generic cuspidal automorphic representations τ and π from the local representations τ_0 and π_0 . Throughout this section, we impose no restriction on p .

3.1. Proposition. *Let $(F, \tau_0, \pi_0, \psi_0) \in \mathfrak{ls}(p, \mathbf{G}_m, \mathbf{G}_n)$ be supercuspidal. Then, there exists a $(k, \tau, \pi, \psi) \in \mathcal{LS}(p, \mathbf{G}_m, \mathbf{G}_n)$ and a set of places $S = \{v_0, v_1, v_2\}$ of k such that*

- (i) $k_{v_0} \simeq F$;
- (ii) $\tau_{v_0} \simeq \tau_0$ and $\pi_{v_0} \simeq \pi_0$;
- (ii) τ_v is an unramified principal series for $v \notin \{v_0, v_1\}$;
- (iv) π_v is an unramified principal series for $v \notin \{v_0, v_2\}$.

3.2. Uniqueness for $\mathfrak{ls}(p, \mathrm{GL}_1, \mathbf{G}_n)$. Let γ be a rule on $\mathfrak{ls}(p, \mathrm{GL}_1, \mathbf{G}_n)$ which assigns to every $(F, \chi, \pi, \psi) \in \mathfrak{ls}(p, \mathrm{GL}_1, \mathbf{G}_n)$ a rational function on q_F^{-s} satisfying properties (i)-(vii). Using property (iv), we can reduce to the case when π is supercuspidal.

Given $(F, \chi_0, \pi_0, \psi_0) \in \mathfrak{ls}(p, \mathrm{GL}_1, \mathbf{G}_n)$ supercuspidal we can lift it to a global $(k, \chi, \pi, \psi) \in \mathcal{LS}(p, \mathrm{GL}_1, \mathbf{G}_n)$ where π_v is an unramified principal series for $v \notin \{v_0, v_1\}$ as in Proposition 3.1. However, in this situation we can take a character χ_{v_0} of $k_{v_0}^\times$ which is isomorphic to χ_0 and a highly ramified character χ_{v_2} of $k_{v_2}^\times$. We then apply the Grundwald-Wang theorem of class field theory [1], in order to lift χ_{v_0} and χ_{v_2} to a grössencharakter $\chi : k^\times \backslash \mathbb{A}_k^\times \rightarrow \mathbb{C}^\times$. From properties (i) and (ii) we have that

$$\gamma(s, \chi_0 \times \pi_0, \psi_0) = \gamma(s, \chi_{v_0} \times \pi_{v_0}, \psi_{v_0}),$$

and we can assume ψ_{v_0} is obtained from ψ_0 using property (v) if necessary.

For every $v \notin \{v_0, v_2\}$, we have that

$$\text{ind}_{B_n}^{G_n}(\mu_{1,v} \otimes \cdots \otimes \mu_{n,v}),$$

where $\mu_{1,v}, \dots, \mu_{n,v}$ are unramified characters. If $\mathbf{G}_n = \text{SO}_{2n}$ or SO_{2n+1} , then

$$(3.1) \quad \gamma(s, \chi_v \times \pi, \psi) = \prod_{i=1}^n \gamma(s, \chi_v \mu_{i,v}, \psi) \gamma(s, \chi_v \mu_{i,v}^{-1}).$$

And, if $\mathbf{G}_n = \text{Sp}_{2n}$, then

$$(3.2) \quad \gamma(s, \chi_v \times \pi, \psi) = \gamma(s, \chi_v, \psi) \prod_{i=1}^n \gamma(s, \chi_v \mu_{i,v}, \psi) \gamma(s, \chi_v \mu_{i,v}^{-1}).$$

The γ -factors on the right hand side of the previous two equations are abelian γ -factors of class field theory. Hence, the rule γ is uniquely determined at these places.

At the place v_2 , we let ξ_1, \dots, ξ_n , be characters of $\text{GL}_1(F)$ such that the restriction of $\xi_1 \otimes \cdots \otimes \xi_n$ to the center of G_n agrees with the central character ω_π of π . Let τ_{v_2} be the generic constituent of

$$\text{ind}_{B_n}^{G_n}(\xi_1 \otimes \cdots \otimes \xi_n).$$

Since we have χ_{v_2} sufficiently ramified we can use property (vi) to obtain

$$\gamma(s, \chi_{v_2} \times \pi_{v_2}, \psi_{v_2}) = \gamma(s, \chi_{v_2} \times \tau_{v_2}, \psi_{v_2}).$$

Then, using multiplicativity, $\gamma(s, \chi_{v_2} \times \tau_{v_2}, \psi_{v_2})$ can be written as a product of abelian γ -factors similar to equations (3.1) and (3.2) above. Any system of γ -factors satisfying properties (i)-(vi) of the Theorem gives the same result at v_2 .

Now, let S be a finite set of places of k including v_0 and such that χ_v , π_v , and ψ_v are unramified for $v \notin S$. Then, property (vii) gives

$$(3.3) \quad L^S(s, \tau \times \pi) = \gamma(s, \tau_{v_0} \times \pi_{v_0}, \psi_{v_0}) \prod_{v \in S - \{v_0\}} \gamma(s, \tau_v \times \pi_v, \psi_v) L^S(1-s, \tilde{\tau}_v \times \tilde{\pi}_v).$$

Since γ -factors are determined for every $v \notin S - \{v_0\}$ by the above discussion, we can conclude that $\gamma(s, \tau_{v_0} \times \pi_{v_0}, \psi_{v_0})$ is completely determined by equation (3.3).

3.3. Uniqueness in general. Let γ be a rule on $\mathfrak{Is}(p)$ which assigns to every quadruple $(F, \tau, \pi, \psi) \in \mathfrak{Is}(p)$ a rational function on q_F^{-s} satisfying properties (i)-(vii). Using property (iv), we can reduce to the supercuspidal case.

Take a fixed supercuspidal $(F, \tau_0, \pi_0, \psi_0) \in \mathfrak{Is}(p, \mathbf{G}_m, \mathbf{G}_n)$ and lift it to a global $(k, \tau, \pi, \psi) \in \mathcal{LS}(p, \mathbf{G}_m, \mathbf{G}_n)$ via Proposition 3.1, properties (i), (ii) and (v). Let \mathbf{B}_m (resp. \mathbf{B}_n) be the Borel subgroup of \mathbf{G}_m (resp. \mathbf{G}_n) of upper triangular matrices. For every $v \notin \{v_0, v_1\}$, let $\chi_{1,v}, \dots, \chi_{m,v}$ be unramified characters of $\text{GL}_1(k_v)$ such that τ_v occurs as a subrepresentation of the unitarily induced representation

$$\text{ind}_{B_m}^{G_m}(\chi_{1,v} \otimes \cdots \otimes \chi_{m,v}).$$

For every $v \notin \{v_0, v_2\}$, let $\mu_{1,v}, \dots, \mu_{n,v}$ be unramified characters of $\text{GL}_1(k_v)$ such that π_v occurs as a subrepresentation of the unitarily induced representation

$$\text{ind}_{B_n}^{G_n}(\mu_{1,v} \otimes \cdots \otimes \mu_{n,v}).$$

Take $v \notin S$, then both τ_v and π_v are unramified. We can in then use properties (iii) and (iv) to show that:

(a) If both \mathbf{G}_m and \mathbf{G}_n are classical groups, then

$$\begin{aligned} \gamma(s, \tau_v \times \pi_v, \psi_v) &= \prod_{i=1}^d \gamma(s, \chi_{i,v} \mu_{0,v}, \psi_v) \gamma(s, \chi_{i,v}^{-1} \mu_{0,v}, \psi_v) \prod_{j=1}^e \gamma(s, \chi_{0,v} \mu_{j,v}, \psi_v) \gamma(s, \chi_{0,v} \mu_{j,v}^{-1}, \psi_v) \\ &\times \prod_{1 \leq h \leq d, 1 \leq l \leq e} \gamma(s, \chi_{h,v} \mu_{l,v}, \psi_v) \gamma(s, \chi_{h,v} \mu_{l,v}^{-1}, \psi_v) \gamma(s, \chi_{h,v}^{-1} \mu_{l,v}, \psi_v) \gamma(s, \chi_{h,v}^{-1} \mu_{l,v}^{-1}, \psi_v), \end{aligned}$$

where we take $\gamma(s, \chi_{0,v} \mu_{j,v}, \psi_v)$ and $\gamma(s, \chi_{0,v} \mu_{j,v}^{-1}, \psi_v)$ (resp. $\gamma(s, \chi_{i,v} \mu_{0,v}, \psi_v)$ and $\gamma(s, \chi_{i,v}^{-1} \mu_{0,v}, \psi_v)$) to be trivial if \mathbf{G}_m (resp. \mathbf{G}_n) is a special orthogonal group; and, we take $\chi_0 = 1$ (resp. $\mu_0 = 1$) if \mathbf{G}_m (resp. \mathbf{G}_n) is symplectic.

(b) If $\mathbf{G}_m = \mathrm{GL}_m$ and \mathbf{G}_n is a classical group, then

$$\gamma(s, \tau_v \times \pi_v, \psi_v) = \prod_{i=0}^d \gamma(s, \chi_{i,v} \mu_{0,v}, \psi_v) \times \prod_{i=1}^d \prod_{j=1}^e \gamma(s, \chi_{i,v} \mu_{j,v}, \psi_v) \gamma(s, \chi_{i,v} \mu_{j,v}^{-1}, \psi_v),$$

where we take $\gamma(s, \chi_{j,v} \mu_{0,v}, \psi_v)$ to be trivial if \mathbf{G}_n is a special orthogonal group; and, we take $\mu_0 = 1$ if \mathbf{G}_n is a symplectic group.

(c) If $\mathbf{G}_m = \mathrm{GL}_m$ and $\mathbf{G}_n = \mathrm{GL}_n$, then

$$\gamma(s, \tau_v \times \pi_v, \psi_v) = \prod_{i,j} \gamma(s, \chi_{i,v} \mu_{j,v}, \psi_v).$$

Any system of γ -factors satisfying properties (i)-(iv) gives $\gamma(s, \tau_v \times \pi_v, \psi_v)$, $v \notin S$, as a product of abelian γ -factors of Tate's thesis as above. Hence, it is uniquely determined at these places.

The remaining possibility, at either v_1 or v_2 , is that one representation is unramified while the other remains arbitrary. For concreteness, assume τ_{v_1} and π_{v_2} are unramified while τ_{v_2} and π_{v_1} remain arbitrary. Then τ_{v_1} and π_{v_2} are constituents of representations via unitary parabolic induction from a product of unramified characters as before. The multiplicativity property of γ -factors gives $\gamma(s, \tau_{v_1} \times \pi_{v_1}, \psi_{v_1})$ as a product of γ -factors of the form $\gamma(s, \chi_{v_1} \times \pi_{v_1}, \psi_{v_1})$, where χ_{v_1} is a character of $\mathrm{GL}_1(k_{v_1})$. Similarly, multiplicativity gives an expression for $\gamma(s, \tau_{v_2} \times \pi_{v_2}, \psi_{v_2})$ as a product of γ -factors of the form $\gamma(s, \tau_{v_2} \times \mu_{v_2}, \psi_{v_2})$, where μ_{v_2} is a character of $\mathrm{GL}_1(k_{v_2})$. In these cases, γ -factors are uniquely determined as shown in § 3.2, where property (vi) is used.

At places where ψ_v may be ramified, we can use property (v) to compute γ -factors with respect to an unramified character. The functional equation for γ -factors gives,

$$L^{S'}(s, \tau \times \pi) = \gamma(s, \tau_{v_0} \times \pi_{v_0}, \psi_{v_0}) \prod_{v \in S' - \{v_0\}} \gamma(s, \tau_v \times \pi_v, \psi_v) L^{S'}(1-s, \tilde{\tau}_v \times \tilde{\pi}_v).$$

Since partial L -functions are uniquely determined, and we have shown that γ -factors are uniquely determined at places other than v_0 , we can conclude that $\gamma(s, \tau_{v_0} \times \pi_{v_0}, \psi_{v_0})$ is uniquely determined.

4. EXTENDED L -FUNCTIONS AND ROOT NUMBERS

We now turn towards the defining properties of L -functions and ε -factors. We assume that $p \neq 2$, which is necessary to study the case $\mathfrak{ls}(p, \mathbf{G}_m, \mathbf{G}_n)$, when \mathbf{G}_m and \mathbf{G}_n are both classical groups.

4.1. Axioms for a local system of L -functions and root numbers. With a system of γ -factors on $\mathfrak{Is}(p)$ satisfying properties (i)-(vii), we can proceed to define rational functions $L(s, \tau \times \pi)$ and monomials $\varepsilon(s, \tau \times \pi, \psi)$ on the variable q_F^{-s} for every $(F, \tau, \pi, \psi) \in \mathfrak{Is}(p)$.

- (viii) (Tempered L -functions). *For $(F, \tau, \psi, \psi) \in \mathfrak{Is}(p)$ tempered, let $P_{\tau \times \pi}(t)$ be the unique polynomial with $P_{\tau \times \pi}(0) = 1$ and such that $P_{\tau \times \pi}(q_F^{-s})$ is the numerator of $\gamma(s, \tau \times \pi, \psi)$. Then*

$$L(s, \tau \times \pi) = P_{\tau \times \pi}(q^{-s})^{-1}.$$

- (ix) (Holomorphy of tempered L -functions). *Let $(F, \tau, \pi, \psi) \in \mathfrak{Is}(p)$ be tempered. Then, $L(s, \tau \times \pi)$ is holomorphic and non-zero for $\Re(s) > 0$.*
(x) (Tempered ε -factors). *Let $(F, \tau, \pi, \psi) \in \mathfrak{Is}(p)$ be tempered, then*

$$\varepsilon(s, \tau \times \pi, \psi) = \gamma(s, \tau \times \pi, \psi) \frac{L(s, \tau \times \pi)}{L(1-s, \tilde{\tau} \times \tilde{\pi})}.$$

- (xi) (Twists by unramified characters). *Let $(F, \tau, \pi, \psi) \in \mathfrak{Is}(p, \mathrm{GL}_m, \mathbf{G}_n)$, then*

$$\begin{aligned} L(s + s_0, \tau \times \pi) &= L(s, |\det(\cdot)|_F^{s_0} \tau \times \pi) \\ \varepsilon(s + s_0, \tau \times \pi, \psi) &= \varepsilon(s, |\det(\cdot)|_F^{s_0} \tau \times \pi, \psi). \end{aligned}$$

- (xii) (Multiplicativity). *Let $(F, \tau_i, \pi_j, \psi) \in \mathfrak{Is}(p, \mathrm{GL}_{m_i}, \mathrm{GL}_{n_j})$, $1 \leq i \leq d$, $1 \leq j \leq e$, be quasi-tempered, and let $(F, \tau_0, \pi_0, \psi) \in \mathfrak{Is}(p, \mathbf{G}_{m_0}, \mathbf{G}_{n_0})$ be tempered. Let $m = \sum m_i$ and $n = \sum n_j$. Let \mathbf{G}_m and \mathbf{G}_n be of the same type as \mathbf{G}_{m_0} and \mathbf{G}_{n_0} . Let \mathbf{P}_m and \mathbf{P}_n be parabolic subgroups of \mathbf{G}_m and \mathbf{G}_n with Levi subgroups $\mathbf{M}_m \simeq \prod_{i=1}^d \mathrm{GL}_{m_i} \times \mathbf{G}_{m_0}$ and $\mathbf{M}_n \simeq \prod_{j=1}^e \mathrm{GL}_{n_j} \times \mathbf{G}_{n_0}$. Suppose that $(F, \tau, \pi, \psi) \in \mathfrak{Is}(p)$ is such that τ is the generic constituent of*

$$\mathrm{ind}_{P_m}^{G_m}(\tau_1 \otimes \cdots \otimes \tau_d \otimes \tau_0),$$

and π is the generic constituent of

$$\mathrm{ind}_{P_n}^{G_n}(\pi_1 \otimes \cdots \otimes \pi_e \otimes \pi_0).$$

When $m_0 = 0$ (resp. $n_0 = 0$) we make the following conventions: take τ_0 (resp. π_0) to be the trivial character of $\mathrm{GL}_1(F)$ if \mathbf{G}_m (resp. \mathbf{G}_n) is a symplectic group; in all other cases, we interpret local factors involving τ_0 (resp. π_0) to be trivial.

- (xii.a) *If both \mathbf{G}_m and \mathbf{G}_n are classical groups, then*

$$\begin{aligned} L(s, \tau \times \pi) &= L(s, \tau_0 \times \pi_0) \\ &\times \prod_{i=1}^d L(s, \tau_i \times \pi_0) L(s, \tilde{\tau}_i \times \pi_0) \prod_{j=1}^e L(s, \tau_0 \times \pi_j) L(s, \tau_0 \times \tilde{\pi}_j) \\ &\times \prod_{i=1}^d \prod_{j=1}^e L(s, \tau_i \times \pi_j) L(s, \tilde{\tau}_i \times \pi_j) L(s, \tau_i \times \tilde{\pi}_j) L(s, \tilde{\tau}_i \times \tilde{\pi}_j), \end{aligned}$$

and local root numbers satisfy

$$\begin{aligned} \varepsilon(s, \tau \times \pi, \psi) &= \varepsilon(s, \tau_0 \times \pi_0, \psi) \\ &\times \prod_{i=1}^d \varepsilon(s, \tau_i \times \pi_0, \psi) \varepsilon(s, \tilde{\tau}_i \times \pi_0, \psi) \prod_{j=1}^e \varepsilon(s, \tau_0 \times \pi_j, \psi) \varepsilon(s, \tau_0 \times \tilde{\pi}_j, \psi) \\ &\times \prod_{i=1}^d \prod_{j=1}^e \varepsilon(s, \tau_i \times \pi_j, \psi) \varepsilon(s, \tilde{\tau}_i \times \pi_j, \psi) \varepsilon(s, \tau_i \times \tilde{\pi}_j, \psi) \varepsilon(s, \tilde{\tau}_i \times \tilde{\pi}_j, \psi). \end{aligned}$$

(xii.b) If $\mathbf{G}_m = \mathrm{GL}_m$ and \mathbf{G}_n is a classical group, then

$$L(s, \tau \times \pi) = \prod_{i=0}^d L(s, \tau_i \times \pi_0) \prod_{i=1}^d \prod_{j=1}^e L(s, \tau_i \times \pi_j) L(s, \tau_i \times \tilde{\pi}_j),$$

and

$$\varepsilon(s, \tau \times \pi, \psi) = \prod_{i=0}^d \varepsilon(s, \tau_i \times \pi_0, \psi) \prod_{i=1}^d \prod_{j=1}^e \varepsilon(s, \tau_i \times \pi_j, \psi) \varepsilon(s, \tau_i \times \tilde{\pi}_j, \psi).$$

(xii.c) If $\mathbf{G}_m = \mathrm{GL}_m$ and $\mathbf{G}_n = \mathrm{GL}_n$, then

$$L(s, \tau \times \pi) = \prod_{i=0}^d \prod_{j=0}^e L(s, \tau_i \times \pi_i),$$

and

$$\varepsilon(s, \tau \times \pi, \psi) = \prod_{i=0}^d \prod_{j=0}^e \varepsilon(s, \tau_i \times \pi_j, \psi).$$

4.2. Additional properties of γ -factors. The following properties are satisfied by any system of γ -factors on $\mathfrak{Is}(p)$ satisfying properties (i)-(vii).

(xiii) (Local functional equation). Let $(F, \tau, \pi, \psi) \in \mathfrak{Is}(p)$, then

$$\gamma(s, \tau \times \pi, \psi) \gamma(1 - s, \tilde{\tau} \times \tilde{\pi}, \overline{\psi}) = 1.$$

(xiv) (Twists by unramified characters). Let $(F, \tau, \pi, \psi) \in \mathfrak{Is}(p, \mathrm{GL}_m, \mathbf{G}_n)$, then

$$\gamma(s + s_0, \tau \times \pi, \psi) = \gamma(s, |\det(\cdot)|_F^{s_0} \tau \times \pi, \psi).$$

(xv) (Commutativity). Let $(F, \tau, \pi, \psi) \in \mathfrak{Is}(p)$, then

$$\gamma(s, \tau \times \pi, \psi) = \gamma(s, \pi \times \tau, \psi).$$

We now give a proof of property (xiii) following the proof of uniqueness given in § 3. Notice that it is a property of abelian γ -factors: if $(F, \chi, \mu, \psi) \in \mathfrak{Is}(p, \mathrm{GL}_1, \mathrm{GL}_1)$, then

$$\gamma(s, \chi \mu, \psi) \gamma(1 - s, \chi^{-1} \mu^{-1}, \overline{\psi}) = 1,$$

see for example equation (1.3) of [11]. First, we prove the local functional equation for the case of $\mathfrak{Is}(p, \mathrm{GL}_1, \mathbf{G}_n)$. We can reduce to the supercuspidal case via multiplicativity. Lift a supercuspidal $(F, \chi_0, \pi_0, \psi_0) \in \mathfrak{Is}(p, \mathrm{GL}_1, \mathbf{G}_n)$ to $(k, \chi, \pi, \psi) \in \mathcal{LS}(p, \mathrm{GL}_1, \mathbf{G}_n)$ as in § 3.2. Notice that the method of proof gives an expression for every $\gamma(s, \chi_v \times \pi_v, \psi_v)$, $v \notin \{v_0\}$, as a product of abelian γ -factors. Then, the local functional equation at v_0 follows from the global functional equation applied twice.

Now, in the proof of uniqueness for the general case of § 3.3, let $(k, \tau, \pi, \psi) \in \mathcal{LS}(p, \mathbf{G}_m, \mathbf{G}_n)$ be the global quadruple obtained from the supercuspidal quadruple $(F, \tau_0, \pi_0, \psi_0) \in \mathfrak{Is}(p, \mathbf{G}_m, \mathbf{G}_n)$. The method of proof gives an expression for every $\gamma(s, \tau_v \times \pi_v, \psi_v)$, $v \notin \{v_0\}$, in terms of γ -factors for $\mathfrak{Is}(p, \mathrm{GL}_1, \mathbf{G}_n)$, $\mathfrak{Is}(p, \mathbf{G}_m, \mathrm{GL}_1)$ or $\mathfrak{Is}(p, \mathrm{GL}_1, \mathrm{GL}_1)$. In all of these cases, the local functional equation holds. Hence, it follows at v_0 by applying the global functional equation twice.

We leave the proofs of properties (xiv) and (xv) as exercises.

4.3. Theorem. *It $p \neq 2$, there **exists** a system of local factors on $\mathfrak{Is}(p)$ satisfying properties (i)-(xii). Any system of local factors on $\mathfrak{Is}(p)$ satisfying properties (i)-(xii) is **uniquely** determined.*

Proof. We have already established the existence and uniqueness part of the theorem concerning properties (i)-(vii). We now construct local L -functions and ε -factors.

We first treat the tempered case, where property (viii) is taken as the definition of a local L -function. Notice that the multiplicativity property of γ -factors gives the multiplicativity property of local L -functions in the tempered case.

We now prove property (ix) for the new case of two classical groups \mathbf{G}_m and \mathbf{G}_n . Let $(F, \tau, \pi, \psi) \in \mathfrak{Is}(p, \mathbf{G}_m, \mathbf{G}_n)$ be tempered. The representation τ is a constituent of

$$\mathrm{ind}_{P_m}^{G_m}(\delta_1 \otimes \cdots \otimes \delta_d \otimes \tau_0),$$

as in equation (2.3) with δ_i , $i = 1, \dots, d$, generic discrete series representations of $\mathrm{GL}_{m_i}(F)$, and τ_0 a generic discrete series representation of G_{m_0} . Similarly, π is a constituent of

$$\mathrm{ind}_{P_n}^{G_n}(\rho_1 \otimes \cdots \otimes \rho_e \otimes \pi_0),$$

with ρ_j , $j = 1, \dots, e$, generic discrete series representations of $\mathrm{GL}_{m_j}(F)$, and π_0 a generic discrete series representation of G_{n_0} .

Let T_0 and Π_0 be the local functorial lifts of τ_0 and π_0 given by equation (2.2). Notice that they are self-dual tempered representations of $\mathrm{GL}_{M_0}(F)$ and $\mathrm{GL}_{N_0}(F)$. The local functorial lift T of τ is given by

$$\mathrm{ind}_{P_m}^{G_m}(\delta_1 \otimes \cdots \otimes \delta_d \otimes T_0 \otimes \tilde{\delta}_1 \otimes \cdots \otimes \tilde{\delta}_1)$$

and the lift Π of π is given by

$$\mathrm{ind}_{P_n}^{G_n}(\rho_1 \otimes \cdots \otimes \rho_e \otimes \Pi_0 \otimes \tilde{\rho}_e \otimes \cdots \otimes \tilde{\rho}_1).$$

Then

$$\begin{aligned} L(s, \tau \times \pi) &= L(s, T_0 \times \Pi_0) \prod_{i=1}^d L(s, \delta_i \times \Pi_0) L(s, \tilde{\delta}_i \times \Pi_0) \prod_{j=1}^e L(s, T_0 \times \rho_j) L(s, T_0 \times \tilde{\rho}_j) \\ &\quad \times \prod_{i=1}^d \prod_{j=1}^e L(s, \delta_i \times \rho_j) L(s, \tilde{\delta}_i \times \rho_j) L(s, \delta_i \times \tilde{\rho}_j) L(s, \tilde{\delta}_i \times \tilde{\rho}_j). \end{aligned}$$

Each L -function on the right hand side is a Rankin-Selberg L -function for products of representations of general linear groups, known to be holomorphic for $\Re(s) > 0$. Hence, the extended L -function $L(s, \tau \times \pi)$ is holomorphic for $\Re(s) > 0$. Thus, our local L -functions satisfy property (ix) in the tempered case.

Next, for tempered $(F, \tau, \pi, \psi) \in \mathfrak{Is}(p)$, property (x) is taken as the definition of root numbers. Then $\varepsilon(s, \tau \times \pi, \psi)$ is a monomial in q_F^{-s} . For this, we use the

local functional equation of γ -factors, property (xiii), together with property (ix) to ensure that no cancellations occur on the strip $0 < \Re(s) < 1$.

Having defined local L -functions and root numbers for tempered representations, they are then defined on $\mathfrak{Is}(p)$ with the aid of Langlands classification. In the case of $\mathbf{G}_m = \mathrm{GL}_m$ and $\mathbf{G}_n = \mathrm{GL}_n$, we include a treatment of L -functions and ε -factors in [11] in a self contained manner within the \mathcal{LS} -method. The definition given by equations (7.6) of [*loc. cit.*] is in accordance with the definition of Rankin-Selberg local factors [7]. We will now define extended L -functions and root numbers on $\mathfrak{Is}(p, \mathbf{G}_m, \mathbf{G}_n)$, when both \mathbf{G}_m and \mathbf{G}_n are classical groups. Obtaining explicit defining relations for local L -functions and ε -factors on $\mathfrak{Is}(p, \mathrm{GL}_m, \mathbf{G}_n)$, with \mathbf{G}_n a classical group, is left as an exercise.

Assume that both \mathbf{G}_m and \mathbf{G}_n are classical groups. Let τ be given by equation (2.5)

$$\mathrm{ind}_{P_m}^{G_m}(\tau_1 \nu^{r_1} \otimes \cdots \otimes \tau_d \nu^{r_d} \otimes \tau_0),$$

where each τ_i is a tempered representation of $\mathrm{GL}_{m_i}(F)$ and τ_0 is a generic tempered representation of G_{m_0} . With the appropriate modifications if τ is given by equation (2.6). Similarly, π is given by

$$\mathrm{ind}_{P_n}^{G_n}(\pi_1 \nu^{t_1} \otimes \cdots \otimes \pi_e \nu^{t_e} \otimes \pi_0),$$

where each π_i is a tempered representation of $\mathrm{GL}_{n_i}(F)$ and π_0 is a generic tempered representation of G_{n_0} . With appropriate modifications if π is given by equation (2.6).

Let $(F, \tau, \pi, \psi) \in \mathfrak{Is}(p)$, then we define local L -functions by

$$\begin{aligned} L(s, \tau \times \pi) &= L(s, \tau_0 \times \pi_0) \\ &\times \prod_{i=1}^d L(s + r_i, \tau_i \times \pi_0) L(s - r_i, \tilde{\tau}_i \times \pi_0) \prod_{j=1}^e L(s + t_j, \tau_0 \times \pi_j) L(s - t_j, \tau_0 \times \tilde{\pi}_j) \\ &\times \prod_{i=1}^d \prod_{j=1}^e L(s + r_i + t_j, \tau_i \times \pi_j) L(s - r_i + t_j, \tilde{\tau}_i \times \pi_j) \\ &\times L(s + r_i - t_j, \tau_i \times \tilde{\pi}_j) L(s - r_i - t_j, \tilde{\tau}_i \times \tilde{\pi}_j), \end{aligned}$$

and local root numbers by

$$\begin{aligned} \varepsilon(s, \tau \times \pi, \psi) &= \varepsilon(s, \tau_0 \times \pi_0, \psi) \\ &\times \prod_{i=1}^d \varepsilon(s + r_i, \tau_i \times \pi_0, \psi) \varepsilon(s - r_i, \tilde{\tau}_i \times \pi_0, \psi) \prod_{j=1}^e \varepsilon(s + t_j, \tau_0 \times \pi_j, \psi) \varepsilon(s - t_j, \tau_0 \times \tilde{\pi}_j, \psi) \\ &\times \prod_{i=1}^d \prod_{j=1}^e \varepsilon(s + r_i + t_j, \tau_i \times \pi_j, \psi) \varepsilon(s - r_i + t_j, \tilde{\tau}_i \times \pi_j, \psi) \\ &\times \varepsilon(s + r_i - t_j, \tau_i \times \tilde{\pi}_j, \psi) \varepsilon(s - r_i - t_j, \tilde{\tau}_i \times \tilde{\pi}_j, \psi). \end{aligned}$$

It is now possible to show that our construction is compatible with properties (xi) and (xii). Notice that the definition of local L -functions and root numbers is based on an extended system of γ -factors, and only uses special cases of properties (viii)-(xii). Hence, any system of extended γ -factors, local L -functions and root numbers satisfying properties (i)-(xii) is uniquely determined.

4.4. Theorem. *Assume that $p \neq 2$. For every $(k, \tau, \pi, \psi) \in \mathcal{LS}(p)$, let*

$$(4.1) \quad L(s, \tau \times \pi) = \prod_v L(s, \tau_v \times \pi_v), \quad \varepsilon(s, \tau \times \pi) = \prod_v \varepsilon(s, \tau_v \times \pi_v, \psi_v).$$

Automorphic L -functions satisfy the following properties:

- (i) (Rationality). *$L(s, \tau \times \pi)$ converges on a right half plane and has a meromorphic continuation to a rational function on q^{-s} .*
- (ii) (Functional equation). *$L(s, \tau \times \pi) = \varepsilon(s, \tau \times \pi) L(1-s, \tilde{\tau} \times \tilde{\pi})$.*
- (iii) (Riemann Hypothesis) *The zeros of $L(s, \tau \times \pi)$ are contained in the line $\Re(s) = 1/2$.*

Proof. First, given $(F, \tau, \pi, \psi) \in \mathfrak{ls}(p, \mathrm{GL}_m, \mathbf{G}_n)$, we know that partial L -functions $L^S(s, \tau \times \pi)$ converge on a right half plane. The rationality argument of Harder for Eisenstein series [2], allows us to give an automorphic forms proof that $L^S(s, \tau \times \pi)$ is a rational function on q^{-s} , Corollary 6.6 of [10]. Now, notice that each local L -function in the product

$$\prod_{v \in S} L(s, \tau_v \times \pi_v)$$

is a rational function on $q_v^{-s} = q^{-\deg(v)s}$. Hence, property (i) follows for completed automorphic L -functions in this case. Also, the definition of local L -functions and ε -factors at the places $v \in S$ can be incorporated into the functional equation satisfied by γ -factors in order to obtain property (ii) for global L -functions and ε -factors on $\mathcal{LS}(p, \mathrm{GL}_m, \mathbf{G}_n)$.

Next, we treat the case of $\mathcal{LS}(p, \mathbf{G}_m, \mathbf{G}_n)$, with both \mathbf{G}_m and \mathbf{G}_n classical groups. Let $(k, \tau, \pi, \psi) \in \mathcal{LS}(p, \mathbf{G}_m, \mathbf{G}_n)$. The functorial lift of Theorem 9.1 of [10] is compatible with the local functorial lift at every place v of k . The construction of the local lift is reviewed in § 2.4. Equation (2.9) enables us to write the global lifts T of τ and Π of π as isobaric sums

$$(4.2) \quad T = T_1 \boxplus \cdots \boxplus T_d \text{ and } \Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_e.$$

We have that

$$L^S(s, \tau \times \pi) = L^S(s, T \times \Pi),$$

which converge on a right half plane and have a meromorphic continuation to a rational function on q^{-s} . At the remaining places, extended L -functions $L(s, \tau_v \times \pi_v)$ are rational on q_v^{-s} . Hence, the completed automorphic L -function $L(s, \tau \times \pi)$ satisfies property (i). The way local factors are defined can be coupled with the functional equation satisfied by extended γ -factors in order to establish property (ii) for automorphic L -functions on $\mathcal{LS}(p, \mathbf{G}_m, \mathbf{G}_n)$.

Finally, let $(k, \tau, \pi, \psi) \in \mathcal{LS}(p, \mathbf{G}_m, \mathbf{G}_n)$. If both \mathbf{G}_m and \mathbf{G}_n are classical groups, take T and Π to be the global functorial lifts of τ and π of equation (4.2). If $\mathbf{G}_m = \mathrm{GL}_m$, we just take $T = \tau$. Then

$$L(s, \tau \times \pi) = L(s, T \times \Pi) = \prod_{i,j} L(s, T_i \times \Pi_j),$$

where $(k, T_i, \Pi_j, \psi) \in \mathcal{LS}(p, \mathrm{GL}_{m_i}, \mathrm{GL}_{n_j})$, for $1 \leq i \leq d$, $1 \leq j \leq e$. Thanks to the work of Lafforgue on the global Langlands conjecture for GL_N over function fields, each Rankin-Selberg L -function $L(s, T_i \times \Pi_j)$ satisfies the Riemann Hypothesis (see Théorème VI.10(ii) of [8]). Hence, so does $L(s, \tau \times \pi)$. We conclude that automorphic L -functions on $\mathcal{LS}(p)$ satisfy the Riemann Hypothesis.

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